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HITCHIN-MOCHIZUKI MORPHISM, OPERS AND FROBENIUS-DESTABILIZED VECTOR BUNDLES OVER CURVES

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ABSTRACT. Let X be a smooth projective curve of genus $g \geq 2$ defined over an algebraically closed field k of characteristic $p > 0$. For p sufficiently large (explicitly given in terms of r, g) we construct an atlas for the locus of all Frobenius-destabilized bundles (i.e. we construct all Frobenius-destabilized bundles of degree zero up to isomorphism). This is done by exhibiting a surjective morphism from a certain Quot-scheme onto the locus of stable Frobenius-destabilized bundles. Further we show that there is a bijective correspondence between the set of stable vector bundles E over X such that the pull-back $F^*(E)$ under the Frobenius morphism of X has maximal Harder-Narasimhan polygon and the set of opers having zero p-curvature. We also show that, after fixing the determinant, these sets are finite, which enables us to derive the dimension of certain Quot-schemes and certain loci of stable Frobenius-destabilized vector bundles over X. The finiteness is proved by studying the properties of the Hitchin-Mochizuki morphism. In particular we prove a generalization of a result of Mochizuki to higher ranks.

1. Introduction

1.1. **The statement of the results.** Let X be a smooth projective curve of genus $g \geq 2$ defined over an algebraically closed field k of characteristic $p > 0$. One of the interesting features of vector bundles in positive characteristic is the existence of semistable vector bundles E over X such that their pull-back $F^*(E)$ under the absolute Frobenius morphism $F : X \to X$ is no longer semistable. This phenomenon also occurs over base varieties of arbitrary dimension and is partly responsible for the many difficulties arising in the construction and the study of moduli spaces of principal G-bundles in positive characteristic. We refer to the recent survey [La] for an account of the developments in this field.

Let us consider the coarse moduli space $\mathcal{M}(r)$ of S-equivalence classes of semistable vector bundles of rank r and degree 0 over a curve X and denote by $\mathcal{J}(r)$ the closed subvariety of $\mathcal{M}(r)$ parameterizing semistable bundles E such that $F^*(E)$ is not semistable. For arbitrary r, g, p, besides their non-emptiness (see [LasP2]), not much is known about the loci $\mathcal{J}(r)$. For example, their dimension and their irreducibility are only known in special cases or for small values of r, p and g; see e.g. [D], [J0], [JRXY], [J2], [LanP], [LasP2],[Mo1], [Mo2], [O1, O2], [Su2]. Following [JRXY] one associates to a stable bundle $E \in \mathcal{J}(r)$ the Harder-Narasimhan polygon of the bundle $F^*(E)$ and defines in that way [Sha] a natural stratification on the stable[1] locus $\mathcal{J}^s(r) \subset \mathcal{J}(r)$. Thus the fundamental question which arises is: what is the geometry of the locus $\mathcal{J}(r)$ and the stratification it carries. Before proceeding further we would like to recall some notions. A well-known theorem of Carter's (see section 2.1.3) says that there is a one-to-one correspondence between vector bundles E over X and local systems (V, ∇) having p-curvature $\psi(V, \nabla)$ zero, which is given by the two mappings

$$E \mapsto (F^*(E), \nabla^{can}) \qquad \text{and} \qquad (V, \nabla) \mapsto V^{\nabla} = E.$$

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[1]We note that the Harder-Narasimhan polygon of $F^*(E)$ may vary when E varies in an S-equivalence class.

Here V^∇ denotes the sheaf of ∇-invariant sections and ∇^{can} the canonical connection. An important class of local systems (in characteristic zero) was studied by Beilinson and Drinfeld in their fundamental work on the geometric Langlands program [BD1]. These local systems are called opers and they play a fundamental role in the geometric Langlands program.

An *oper* is a triple $(V, \nabla, V_\bullet)$ (see Definition 3.1.1) consisting of a vector bundle V over X, a connection ∇ on V and a flag $V_\bullet$ satisfying some conditions. In their original paper [BD1] (see also [BD2]) the authors define opers (with complete flags) over the complex numbers and identify them with certain differential operators between line bundles. We note that over a smooth projective curve X the underlying vector bundle V of an oper as defined in [BD1] is constant up to tensor product by a line bundle: in the rank two case the bundle V is also called the *Gunning bundle* $\mathcal{G}$ (see [Gu], [Mo1]) and is given by the unique non-split extension of θ^{-1} by θ for a theta-characteristic θ of the curve X. For higher rank r the bundle V equals the symmetric power $\mathrm{Sym}^{r-1}(\mathcal{G})$ up to tensor product by a line bundle. In particular, the bundle V is non-semistable and we shall denote by $\mathscr{P}_r^{oper}$ its Harder-Narasimhan polygon. Opers of rank two appeared in characteristic $p > 0$ in the work of S. Mochizuki (see [Mo1]), where they appeared as indigenous bundles. An oper is *nilpotent* if the underlying connection is nilpotent (of exponent $\leq$ rank of the oper). An oper is *dormant* if the underlying connection has p-curvature zero (this terminology is due to S. Mochizuki). By definition any dormant oper is nilpotent. In [Mo1], Mochizuki proved a foundational result: the scheme of nilpotent, indigenous bundles is finite. This result lies at the center of Mochizuki's p-adic uniformization program. Opers of higher rank in positive characteristic $p > 0$ also appeared in [JRXY]. We will take a slightly more general definition of opers by allowing non-complete flags. With our definition the triple $(F^*(F_*(Q)), \nabla^{can}, V_\bullet)$ associated to any vector bundle Q over X, as introduced in [JRXY] (we note that in [JRXY] this was shown under assumption that $F_*(Q)$ is stable if Q is stable; this restriction was removed in [LanP] for Q of rank one, and more recently in all cases by [Su2]) see Theorem 3.1.6), is an oper, even a dormant oper.

Our first result is the higher rank case of the finiteness result of [Mo1]–that the locus of nilpotent PGL_2-opers is finite (see Theorem 6.1.3):

Theorem 1.1.1. *The scheme* $\mathrm{Nilp}_r(X)$ *is finite.*

We note that Mochizuki allows curves with log structures, but we do not. Theorem 1.1.1 is proved by proving that $\mathrm{Nilp}_r(X)$ is both affine and proper. One gets affineness by constructing $\mathrm{Nilp}_r(X)$ as the fiber over 0 of what we call the *Hitchin-Mochizuki morphism* (see section 3.3)

$$\mathrm{HM} : \mathfrak{Op}_{\mathrm{PGL}(r)}(X) \longrightarrow \bigoplus_{i=2}^{r} H^0(X, (\Omega^1_X)^{\otimes i}), \qquad (V, \nabla, V_\bullet) \mapsto [\mathrm{Char}\ \psi(V,\nabla)]^{\frac{1}{p}},$$

which associates to an oper $(V, \nabla, V_\bullet)$ the p-th root of the coefficients of the characteristic polynomial of the p-curvature. That the scheme of PGL_r-opers, denoted here by $\mathfrak{Op}_{PGL(r)}$ is affine is due to Beilinson-Drinfeld (see [BD1]).

This morphism is a generalization of the morphism, first introduced and studied by S. Mochizuki (see [Mo1, page 1025]) in the rank two case and for families of curves with logarithmic structures, and called the *Verschiebung*. We consider a generalization of this to arbitrary rank and we call it the Hitchin-Mochizuki morphism. Our approach to Theorem 1.1.1 is different from that of [Mo1, Mo2]. The key point is: the Mochizuki's verschiebung is really a Hitchin map with affine source and target. From this optic finiteness is equivalent to properness, and we note that Hitchin maps (all of them) share a key property: properness. Thus one sees from this that finiteness of indigenous bundles is a rather natural consequence of being a fibre of a

Hitchin(-Mochizuki) map. Thus it remains to prove, exactly as is the case with usual Hitchin morphism ([F], [N]), that the Hitchin-Mochizuki morphism HM is proper (Remark 6.1.7). We mention that both sides of the morphism HM are affine and of the same dimension. At any rate, from this point of view we reduce the proof of finiteness to proving a suitable valuative criterion. This is accomplished by showing that the underlying local system of an oper is stable (Proposition 3.4.2) and by proving a semistable reduction theorem for nilpotent opers (Proposition 6.1.4). We note that the finiteness result for indigenous bundles was proved by S. Mochizuki in the rank two case (see [Mo1]) by a different method which does not seem (at the moment) to lend itself to generalization to higher ranks as it uses some rather non-linear properties of p-curvature and some peculiarities of rank 2. The fibre over zero of the Hitchin-Mochizuki morphism consists of nilpotent opers–this is the analogue of the global nilpotent cone of the usual Hitchin morphism. Since dormant opers (i.e. opers with p-curvature zero) are nilpotent, we deduce from Theorem 1.1.1 that the scheme of dormant opers with fixed determinant is finite. In the rank two, genus two case it is possible to compute by various methods the length of this scheme (see [Mo2], [LanP] or [O2]). Our finiteness result Theorem 1.1.1 naturally raises the problem of counting the nilpotent and dormant opers for general r, p and g. In [Mo1] Mochizuki also showed among other results that $\mathrm{Nilp}_2(X)$ is a scheme of length $p^{\dim \mathfrak{Op}_{\mathrm{PGL}(2)}(X)}$. We expect that an analogous result holds for higher ranks as well but we do not pursue this question here.

The key technical tool in the proof of Theorem 1.1.1 is the following result which controls the Harder-Narasimhan polygon of Frobenius destabilized bundles (also see Theorem 5.3.1).

Theorem 1.1.2. *Let (V, ∇) be a semistable local system of degree 0 and rank r over the curve X. Let $V_\bullet^{HN}$ denote the Harder-Narasimhan filtration of V. Then*

(1) *the Harder-Narasimhan polygon $\mathscr{P}_V$ of V is on or below the oper-polygon $\mathscr{P}_r^{oper}$.*
(2) *we have equality $\mathscr{P}_V = \mathscr{P}_r^{oper}$ if and only if $(V, \nabla, V_\bullet^{HN})$ is an oper.*

To better appreciate this result it is best to put in proper perspective. We note that bundles with connections of p-curvature zero are crystals (in the sense of Grothendieck–though not F-crystals) and this theorem is, in a sense, an analog of Mazur's theorem ("Katz' conjecture") on Hodge and Newton polygons of F-crystals see [Maz73]–the oper polygon has integer slopes and plays the role of the Hodge polygon (and if we apply a ninety degree counter clockwise rotation to the polygon plane, then our result says that the rotated Harder-Narasimhan polygon lies on or above the oper-polygon and both have the same endpoints).

As in the case of Mazur's Theorem, our result is equivalent to a list of inequalities for the slopes of the graded pieces of the Harder-Narasimhan filtration. We establish the result by proving that the relevant list of inequalities hold using by induction and by combining and refining some well-known inequalities (see [She], [Su1] or [LasP1]) on the slopes of the successive quotients of $V_\bullet^{HN}$. As a particular case of Theorem 1.1.2 we obtain that, when the semistable bundle E varies, the Harder-Narasimhan polygon of $F^*(E)$ is maximal if and only if $(F^*(E), \nabla^{can}, F^*(E)_\bullet^{HN})$ is a dormant oper.

We now turn to describing our results on the Frobenius instability locus in the moduli space of semi-stable vector bundles on curves. As pointed out in the introduction, the problem of describing the Frobenius locus is one of the least understood aspect of the theory of vector bundles on curves. Our next result, provides a construction of all Frobenius destabilized stable bundles. We state the result in its weakest form:

Theorem 1.1.3. *Let X be a smooth, projective curve of genus $g \geq 2$ over an algebraically closed field k of characteristic $p > 0$. If $p > C(r, g)$, then we have*

(1) *Every stable and Frobenius-destabilized vector bundle* V *of rank* r *and slope* $\mu(V) = \mu$ *over* X *is a subsheaf* $V \hookrightarrow F_*(Q)$ *for some stable vector bundle* Q *of rank* $\mathrm{rk}(Q) < r$ *and* $\mu(Q) < p\mu$.

(2) *Conversely, given a semistable vector bundle* Q *with* $\mathrm{rk}(Q) < r$ *and* $\mu(Q) < p\mu$, *every subsheaf* $V \hookrightarrow F_*(Q)$ *of rank* $\mathrm{rk}(V) = r$ *and slope* $\mu(V) = \mu$ *is semistable and destabilized by Frobenius.*

It should be remarked that, it has been suspected after [JRXY], [LanP], that the every Frobenius destabilized stable bundle V should occur as a subsheaf of $F_*(Q)$ for a suitably chosen Q. The main difficulty in realizing this expectation was the injectivity $V \hookrightarrow F_*(Q)$ for the canonical choice of Q (the one given by the minimal slope quotient of the instability flag of $F^*(V)$). We are able to resolve this difficulty, and it should be remarked that opers play a critical role in this resolution as well, by providing a uniform bound on the slopes of subbundles of $F_*(Q)$ (for any stable Q) (see Proposition 4.2.1). This bound is finer than the bounds which exists in literature (see [LanP],[JRXY] and [J1]). Our method is to use a method of Xiaotao Sun (see [Su2]), especially an important formula (see eq 4.2.3) due to him, relating the slope of a subbundle of $W \subset F_*(Q)$ to the slopes of the graded pieces of the filtration induced by the oper filtration. We show that Sun's formula reduces the problem of obtaining a bound to a numerical optimization problem. We are able to carry out this optimization effectively (for p bigger than an explicit constant depending on the degree and the rank, see the statement of the theorem for details). The final step is to show that every stable Frobenius destabilized bundle V is, in fact, a subsheaf of $F_*(Q)$ for a Q of some rank $\leq \mathrm{rk}(V) - 1$ and $\deg(Q) = -1$. This is the optimal degree allowed from Harder-Narasimhan flag slope considerations and this is accomplished in Theorem 4.1.2.

This paves the way for describing the instability locus $\mathcal{J}(r)$ in terms of a suitable quot scheme. We show that under the assumption that $p > C(r, g)$, where $C(r, g)$ is an explicit constant depending only on r and g, there is a morphism π (see Theorem 4.4.1 for precise statement) from a disjoint union of relative Quot-schemes to $\mathcal{J}(r)$, which surjects onto the stable part of the instability locus $\mathcal{J}^s(r)$. We expect that π is birational over some irreducible component of $\mathcal{J}(r)$, but we are not able to show that for $r > 2$ and we hope to return to it at a later work. We also show (Proposition 5.4.2) that the set of dormant opers equals a relative Quot-scheme

$$\alpha : \mathcal{Q}\mathrm{uot}(r, 0) \to \mathrm{Pic}^{-(r-1)(g-1)}(X)$$

over the Picard variety of line bundles over X of degree $-(r-1)(g-1)$ such that the fiber $\alpha^{-1}(Q)$ over $Q \in \mathrm{Pic}^{-(r-1)(g-1)}(X)$ equals the Quot-scheme $\mathrm{Quot}^{r,0}(F_*(Q))$ parameterizing subsheaves of degree 0 and rank r of the direct image $F_*(Q)$. In particular, since $\mathrm{Quot}^{r,0}(F_*(Q)) \neq \emptyset$ (Proposition 2.3.2), we obtain the existence of dormant opers.

As mentioned earlier, the instability locus $\mathcal{J}(r)$, and especially $\mathcal{J}^s(r)$, is equipped with a stratification, and one would like a concrete description of the instability strata. This is at the moment only understood completely for $p = 2, r = 2$ ([JRXY]), though there are partial results for $p \leq 3, g \leq 3, r \leq 3$, or $p \geq 2g, r \leq 2$, in the references cited at the end of the first paragraph of this paper. Our first result (Theorem 1.1.1) identifies the minimal dimensional stratum of this stratification on $\mathcal{J}^s(r)$. The finiteness result Theorem 1.1.1 provides the following description of the minimal dimensional stratum which corresponds to the highest Harder-Narasimhan polygon: the stratum corresponding to the oper-polygon is zero dimensional. We call this the *operatic locus*. The operatic locus is characterized as the locus of semi-stable bundles whose Frobenius pull-back and its Harder-Narasimhan filtration together with its Cartier connection

and is a dormant oper. The finiteness result Theorem 1.1.1 shows that the operatic locus is finite Theorem 5.4.1. As applications of the relationship between opers and Quot-schemes, we mention the following results. Firstly, for rank two, we show (Theorem 7.1.2) that any irreducible component of $\mathcal{J}(2)$ containing a dormant oper has dimension $3g-4$, which completes some results on $\dim \mathcal{J}(2)$ for general curves due to S. Mochizuki. Secondly, we deduce from Theorem 1.1.1 that $\dim \mathrm{Quot}^{r,0}(F_*(Q)) = 0$ (its expected dimension) for any line bundle Q of degree $-(r-1)(g-1)$.

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Contents

2. Generalities

2.1. The Frobenius morphism and p-curvature.

2.1.1. *Definitions.* Let X be a smooth projective curve of genus $g \geq 2$ defined over an algebraically closed field k of characteristic $p > 0$. Let $F : X \to X$ be the absolute Frobenius morphism of the curve X. We denote by Ω^1_X and T_X the canonical bundle and the tangent bundle of the curve X.

Given a local system (V, ∇) over X, i.e. a pair (V, ∇) consisting of a vector bundle V over X and a connection ∇ on V, we associate (see e.g. [K]) the p-curvature morphism

$$\psi(V, \nabla) : T_X \longrightarrow \mathrm{End}(V), \qquad D \mapsto \nabla(D)^p - \nabla(D^p).$$

Here D denotes a local vector field, D^p its p-th power and $\mathrm{End}(V)$ denotes the sheaf of $\mathcal{O}_X$-linear endomorphisms of V.

2.1.2. *Properties of the p-curvature morphism.* By [K] Proposition 5.2 the p-curvature morphism is $\mathcal{O}_X$-semi-linear, which means that it corresponds to an $\mathcal{O}_X$-linear map, also denoted by $\psi(V,\nabla)$

$$\psi(V,\nabla) : F^*T_X \longrightarrow \mathrm{End}(V). \tag{2.1.1}$$

In the sequel we will always consider p-curvature maps as $\mathcal{O}_X$-linear maps (2.1.1). Given local systems (V,∇_V) and (W,∇_W) over X, we can naturally associate the local systems given by their tensor product $(V\otimes W, \nabla_V\otimes\nabla_W)$ and the determinant $(\det V, \det\nabla_V)$.

Proposition 2.1.2. *We have the following relations on the p-curvature maps*

(i) $\psi(\mathcal{O}_X, d) = 0$, *where* $d : \mathcal{O}_X \to \Omega^1_X$ *is the differentiation operator.*
(ii) $\psi(V\otimes W, \nabla_V\otimes\nabla_W) = \psi(V,\nabla_V)\otimes \mathrm{Id}_W + \mathrm{Id}_V \otimes \psi(W,\nabla_W)$.
(iii) $\psi(\det V, \det\nabla_V) = \mathrm{Tr}\circ\psi(V,\nabla_V)$, *where* $\mathrm{Tr} : \mathrm{End}(V) \longrightarrow \mathcal{O}_X$ *denotes the trace map.*

Proof. Part (i) is trivial. Part (ii) follows from the definition of the p-curvature and the fact that the two terms of $\nabla_{V\otimes W}(D) = \nabla_V(D)\otimes\mathrm{Id}_W + \mathrm{Id}_V\otimes\nabla_W(D)$ commute in the ring $\mathrm{End}_k(V)$ of k-linear endomorphisms of V. As for part (iii), we iterate the formula proved in (ii) in order to obtain an expression for the p-curvature $\psi(V^{\otimes r}, \nabla_{V^{\otimes r}})$ with $r = \mathrm{rk}(V)$ and show that the $F^*\Omega^1_X$-valued endomorphism $\psi(V^{\otimes r}, \nabla_{V^{\otimes r}})$ of $V^{\otimes r}$ induces an endomorphism on the quotient $V^{\otimes r} \longrightarrow \Lambda^r V = \det V$. The formula in (iii) then follows by taking a basis of local sections of V and comparing both sides of the equality. □

Let (L,∇_L) be a local system of rank 1, i.e. L is a line bundle. Let r be an integer not divisible by p. We say that (L,∇_L) is an r-torsion local system if $(L^{\otimes r}, \nabla_{L^{\otimes r}}) = (\mathcal{O}_X, d)$. Note that for an r-torsion line bundle L there exists a unique connection ∇_L on L such that the local system (L,∇_L) is r-torsion.

Proposition 2.1.3. *Let (L,∇_L) be an r-torsion local system. If p does not divide r, then*

$$\psi(L,\nabla_L) = 0.$$

Proof. By Proposition 2.1.2 (i) and (ii) we know that

$$0 = \psi(\mathcal{O}_X, d) = \psi(L^{\otimes r}, \nabla_{L^{\otimes r}}) = r\psi(L,\nabla_L).$$

Hence, if p does not divide r, we obtain the result. □

2.1.3. *Cartier's theorem.* We now state a fundamental property of the p-curvature saying roughly that the tensor $\psi(V,\nabla)$ is the obstruction to descent of the bundle V under the Frobenius map $F : X \to X$.

Theorem 2.1.4 ([K] Theorem 5.1)**.** (1) *Let E be a vector bundle over X. The pull-back $F^*(E)$ under the Frobenius morphism carries a canonical connection ∇^{can}, which satisfies the equality $\psi(F^*(E), \nabla^{can}) = 0$.*
(2) *Given a local system (V,∇) over X, there exists a vector bundle E such that $(V,\nabla) = (F^*(E), \nabla^{can})$ if and only if $\psi(V,\nabla) = 0$.*

2.2. **Direct images under the Frobenius morphism.** Applying the Grothendieck-Riemann-Roch formula to the morphism $F : X \to X$ we obtain the following useful formula for the degree of the direct image of a vector bundle under the Frobenius morphism

Lemma 2.2.1. *Let Q be a vector bundle of rank $q = \mathrm{rk}(Q)$ over X. Then we have*

$$\deg(F_*(Q)) = \deg(Q) + q(p-1)(g-1), \qquad \text{and} \qquad \mu(F_*(Q)) = \frac{\mu(Q)}{p} + \left(1 - \frac{1}{p}\right)(g-1).$$

2.3. The Hirschowitz bound and Quot-schemes.

2.3.1. *Existence of subbundles of given rank and degree.* We will use the following result due to A. Hirschowitz [H] (see also [L]).

Theorem 2.3.1. *Let X be a smooth, projective curve of genus $g \geq 2$. Let V be a vector bundle of rank n and degree d over X. Let m be an integer satisfying $1 \leq m \leq n-1$. Then there exists a subbundle $W \subset V$ of rank m such that*

$$\mu(W) \geq \mu(V) - \left(\frac{n-m}{n}\right)(g-1) - \frac{\varepsilon}{mn},$$

where ε is an integer satisfying $0 \leq \varepsilon \leq n-1$ and

$$\varepsilon + m(n-m)(g-1) \equiv md \mod n.$$

2.3.2. *Non-emptiness of Quot-schemes.* Let Q be a vector bundle of rank q and let r be an integer satisfying $q < r < pq$. We denote by

$$\mathrm{Quot}^{r,0}(F_*(Q))$$

the Quot-scheme parameterizing rank-r subsheaves of degree 0 of the vector bundle $F_*(Q)$.

The following result is an immediate consequence of Theorem 2.3.1.

Proposition 2.3.2. *If $\deg(Q) \geq -(r-q)(g-1)$, then $\mathrm{Quot}^{r,0}(F_*(Q)) \neq \emptyset$.*

Proof. By Theorem 2.3.1 there exists a subsheaf $W \subset F_*(Q)$ with $\mathrm{rk}(W) = r$ such that

$$\mu(W) \geq \mu(F_*(Q)) - \left(\frac{pq-r}{pq}\right)(g-1) - \frac{\varepsilon}{pqr}, \tag{2.3.3}$$

where ϵ is an integer satisfying $0 \leq \varepsilon \leq pq-1$ and

$$\varepsilon + r(pq-r)(g-1) = r\deg(F_*(Q)) \mod pq.$$

Now by Lemma 2.2.1 we have $\deg(F_*(Q)) = \deg(Q) + q(p-1)(g-1)$, so that

$$\begin{aligned}
\varepsilon &= r(\deg(Q) + q(p-1)(g-1)) - r(pq-r)(g-1) \mod pq, \\
&= r\deg(Q) + pqr(g-1) - rq(g-1) - pqr(g-1) + r^2(g-1) \mod pq \\
&= r\deg(Q) - rq(g-1) + r^2(g-1) \mod pq \\
&= r\left[\deg(Q) + (r-q)(g-1)\right] \mod pq
\end{aligned}$$

First we consider the case when $r\left[\deg(Q) + (r-q)(g-1)\right] \leq pq - 1$. We replace ε with $r\left[\deg(Q) + (r-q)(g-1)\right]$ in the inequality (2.3.3) and we obtain $\mu(W) \geq 0$. Thus we can find a subsheaf W of rank r of $F_*(Q)$ with $\deg(W) \geq 0$. If $\deg(W) > 0$, then we take a lower modification V of W to get a subsheaf with $\deg(V) = 0$ and $\mathrm{rk}(V) = r$.

Secondly, we consider the case when $r\left[\deg(Q) + (r-q)(g-1)\right] \geq pq$ or, equivalently $\deg(Q) \geq \frac{pq}{r} - (r-q)(g-1)$. When we combine this inequality and $\frac{-\varepsilon}{pqr} \geq -\frac{1}{r}$, with inequality (2.3.3) we obtain the lower bound $\mu(W) \geq 0$, and we can conclude as above. $\square$

2.3.3. *Dimension estimates for the Quot-scheme.* We recall the following result [Gr], which will be used in Section 7 in the rank two case.

Proposition 2.3.4. *Any irreducible component of* $\mathrm{Quot}^{r,0}(F_*(Q))$ *has dimension at least*

$$r\left[(r-q)(g-1)+\deg(Q)\right].$$

Proof. By [Gr] the dimension at a point $E \in \mathrm{Quot}^{r,0}(F_*(Q))$ is at least $\chi(X, \mathrm{Hom}(E, F_*(Q)/E))$, which is easily computed using Lemma 2.2.1. □

3. Opers

3.1. **Definition and examples of opers and dormant opers.** Opers were introduced by A. Beilinson and V. Drinfeld in [BD1] (see also [BD2]). In this paper, following [JRXY, Section 5], we slightly modify the definition (in [BD1, BD2]).

Definition 3.1.1. *An* oper *over a smooth algebraic curve* X *defined over an algebraically closed field* k *of characteristic* $p > 0$ *is a triple* $(V, \nabla, V_\bullet)$*, where*

(1) V *is a vector bundle over* X*,*
(2) ∇ *is a connection on* V*,*
(3) $V_\bullet : 0 = V_l \subset V_{l-1} \subset \cdots \subset V_1 \subset V_0 = V$ *is a decreasing filtration by subbundles of* V*, called the oper flag.*

These data have to satisfy the following conditions

(1) $\nabla(V_i) \subset V_{i-1} \otimes \Omega^1_X$ *for* $1 \leq i \leq l-1$*,*
(2) *the induced maps* $(V_i/V_{i+1}) \xrightarrow{\nabla} (V_{i-1}/V_i) \otimes \Omega^1_X$ *are isomorphisms for* $1 \leq i \leq l-1$.

Definition 3.1.2. *We say that an oper* $(V, \nabla, V_\bullet)$ *is a* dormant oper *if* $\psi(V, \nabla) = 0$*, i.e. by Cartier's theorem, if the oper* $(V, \nabla, V_\bullet)$ *is of the form* $(F^*(E), \nabla^{can}, V_\bullet)$.

Given an oper $(V, \nabla, V_\bullet)$ we denote by Q the first quotient of $V_\bullet$, i.e.,

$$Q = V_0/V_1.$$

We define the *degree*, *type* and *length* of an oper $(V, \nabla, V_\bullet)$ by

$$\deg(V, \nabla, V_\bullet) = \deg(V), \qquad \mathrm{type}(V, \nabla, V_\bullet) = \mathrm{rk}(Q), \qquad \mathrm{length}(V, \nabla, V_\bullet) = l.$$

The following formulae are immediately deduced from the definition:

$$V_i/V_{i+1} \cong Q \otimes (\Omega^1_X)^{\otimes i} \qquad \text{for } 0 \leq i \leq l-1, \tag{3.1.3}$$

$$\deg(V) = l(\deg(Q) + \mathrm{rk}(Q)(l-1)(g-1)), \qquad \mathrm{rk}(V) = \mathrm{rk}(Q)l. \tag{3.1.4}$$

We observe that the existence of a connection on V implies that $\deg(V)$ is divisible by p. Note that our notion of oper of type 1 corresponds to the notion of $\mathrm{GL}(l)$-oper in [BD1].

Remark 3.1.5. *If the quotient* Q *is semistable, then the oper flag* $V_\bullet$ *coincides with the Harder-Narasimhan filtration of the vector bundle* V.

The following result which combines the results of [JRXY, Section 5] and [Su2] provides the basic example of opers in characteristic $p > 0$.

Theorem 3.1.6. *Let E be any vector bundle over X and let $F : X \to X$ be the absolute Frobenius of X. Then the triple*

$$(V = F^*(F_*(E)), \nabla^{can}, V_\bullet),$$

where $V_\bullet$ is the canonical filtration defined in [JRXY] *section 5.3, is a* dormant oper *of type* $\mathrm{rk}(E)$ *and length p. Moreover, there is an equality $V_0/V_1 = Q = E$.*

3.2. **Recollection of results on opers.** In this section we concentrate on opers of type 1, which we call opers for simplicity. Let $r \geq 2$ be an integer. The results which are mentioned in this section, and which are proved in [BD1] over the complex numbers, are still valid in characteristic $p > 0$ under the assumption that p does not divide r and p odd.

We introduce the algebraic stack (see [BD1] section 3.1) $\mathfrak{Op}_{\mathrm{GL}(r)}(X)$ parameterizing opers of rank r over the curve X. We consider the closed substack $\mathfrak{Op}_{\mathrm{SL}(r)}(X) \hookrightarrow \mathfrak{Op}_{\mathrm{GL}(r)}(X)$ parameterizing opers with fixed trivial determinant, i.e. triples $(V, \nabla, V_\bullet)$ together with isomorphisms $(\det V, \det \nabla_V) \xrightarrow{\sim} (\mathcal{O}_X, d)$, and the stack $\mathfrak{Op}_{\mathrm{PGL}(r)}(X)$ parameterizing rank-r opers up to torsion by rank-1 local systems. Note that the group homomorphism $\mathrm{SL}(r) \to \mathrm{PGL}(r)$ induces a morphism between stacks

$$\mathrm{pr} : \mathfrak{Op}_{\mathrm{SL}(r)}(X) \longrightarrow \mathfrak{Op}_{\mathrm{PGL}(r)}(X).$$

We introduce the vector space

$$W_r = \bigoplus_{i=2}^{r} H^0(X, (\Omega^1_X)^{\otimes i})$$

of dimension $(g-1)(r^2-1)$. In [BD1] section 3.19 one defines an action of the vector space W_r on the algebraic stack $\mathfrak{Op}_{\mathrm{PGL}(r)}(X)$ and, by showing that this action is free and transitive, one obtains

Proposition 3.2.1 ([BD1] Proposition 3.1.10)**.** *The algebraic stack $\mathfrak{Op}_{\mathrm{PGL}(r)}(X)$ is a scheme (non-canonically) isomorphic to the affine space $\mathbb{A}_k^{(g-1)(r^2-1)}$.*

The group $H^1(X, \mu_r)$ of r-torsion line bundles acts on the stack $\mathfrak{Op}_{\mathrm{SL}(r)}(X)$ via tensor product with r-torsion local systems. The set of connected components of $\mathfrak{Op}_{\mathrm{SL}(r)}(X)$ is in one-to-one correspondence with $H^1(X, \mu_r)$.

Given a theta-characteristic θ of X, one defines (see [BD1] section 3.4.2) a section

$$\sigma_\theta : \mathfrak{Op}_{\mathrm{PGL}(r)}(X) \longrightarrow \mathfrak{Op}_{\mathrm{SL}(r)}(X),$$

for the projection pr, i.e. $\mathrm{pr} \circ \sigma_\theta = \mathrm{id}$. We also note that, if r is odd, the section σ_θ does not depend on the theta-characteristic θ.

3.3. **The Hitchin-Mochizuki morphism.** We introduce the algebraic stacks $\mathrm{Loc}_{\mathrm{GL}(r)}$ and $\mathrm{Loc}_{\mathrm{SL}(r)}$ parameterizing rank-r local systems over X respectively rank-r local systems with fixed trivial determinant. We consider the morphism

$$\mathrm{Loc}_{\mathrm{GL}(r)} \longrightarrow \oplus_{i=1}^r H^0(X, F^*(\Omega^1_X)^{\otimes i}), \qquad (V, \nabla) \mapsto \mathrm{Char}(\psi(V, \nabla)), \tag{3.3.1}$$

where $\mathrm{Char}(\psi(V, \nabla))$ denotes the vector in $\oplus_{i=1}^r H^0(X, F^*(\Omega^1_X)^{\otimes i})$ whose components are given by the coefficients of the characteristic polynomial of the p-curvature $\psi(V, \nabla) : V \to V \otimes F^*(\Omega^1_X)$. It is shown in [LasP1] Proposition 3.2 that the components of $\mathrm{Char}(\psi(V, \nabla))$ descend under the Frobenius morphism. This implies that the morphism (3.3.1) factorizes as

$$\mathrm{Loc}_{\mathrm{GL}(r)} \xrightarrow{\Phi} V_r := \oplus_{i=1}^r H^0(X, (\Omega^1_X)^{\otimes i}) \xrightarrow{F^*} \oplus_{i=1}^r H^0(X, F^*(\Omega^1_X)^{\otimes i}).$$

Proposition 3.3.2. *The image of the restriction of the morphism* Φ *to the closed substack* $\mathrm{Loc}_{\mathrm{SL}(r)}$ *is contained in the subspace* $W_r \subset V_r$*.*

Proof. This is an immediate consequence of Proposition 2.1.2 (i) and (iii). □

Given a theta-characteristic θ on X, we consider the composite morphism

$$\mathrm{HM} : \mathfrak{Op}_{\mathrm{PGL}(r)}(X) \xrightarrow{\sigma_\theta} \mathfrak{Op}_{\mathrm{SL}(r)}(X) \longrightarrow \mathrm{Loc}_{\mathrm{SL}(r)} \xrightarrow{\Phi} W_r,$$

which we call the *Hitchin-Mochizuki* morphism. By Proposition 3.2.1 the Hitchin-Mochizuki morphism can be identified with a self-map of the affine space $\mathbb{A}_k^{(g-1)(r^2-1)}$.

Proposition 3.3.3. *The morphism* HM *does not depend on the theta-characteristic* θ*.*

Proof. Given two theta-characteristics θ and θ' on X, we can write $\theta' = \theta \otimes \alpha$ with α a 2-torsion line bundle on X. By [BD1] section 3.4.2 there exists an r-torsion line bundle L (depending on α) such that $\sigma_{\theta'} = T_L \circ \sigma_\theta$, where T_L is the automorphism of $\mathfrak{Op}_{\mathrm{SL}(r)}(X)$ induced by tensor product with the r-torsion local system (L, ∇_L). Now the result follows because, by Proposition 2.1.2 (ii) and by Proposition 2.1.3, we have for any local system (V, ∇_V)

$$\psi(V \otimes L, \nabla_V \otimes \nabla_L) = \psi(V, \nabla_V) + \mathrm{Id}_V \otimes \psi(L, \nabla_L) = \psi(V, \nabla_V).$$

□

3.4. **Semistability.** We recall the notion of semistability, introduced in [Si], for a local system, i.e., a pair (V, ∇), where V is a vector bundle over X and ∇ is a connection on V.

Definition 3.4.1. *We say that the local system* (V, ∇) *is* semistable *(resp.* stable*) if any proper* ∇*-invariant subsheaf* $W \subset V$ *satisfies* $\mu(W) \leq \mu(V)$ *(resp.* $\mu(W) < \mu(V)$*).*

In the particular case of a local system (V, ∇) with $\psi(V, \nabla) = 0$ the semistability condition can be expressed as follows. In that case, by Cartier's theorem (Theorem 2.1.4), the local system (V, ∇) is of the form $(V, \nabla) = (F^*(E), \nabla^{can})$ for some vector bundle E over X. Then, since ∇^{can}-invariant subsheaves of $F^*(E)$ correspond to subsheaves of E, the local system $(F^*(E), \nabla^{can})$ is semistable if and only if E is semistable.

Proposition 3.4.2. *Let* $(V, \nabla, V_\bullet)$ *be an oper of any type with* $Q = V_0/V_1$*. If* Q *is semistable (resp. stable), then the local system* (V, ∇) *is semistable (resp. stable).*

Proof. We will need the following

Lemma 3.4.3. *Let* $(V, \nabla, V_\bullet)$ *be an oper of any type with* $Q = V_0/V_1$ *and let* $W \subset V$ *be a* ∇*-invariant subbundle. We consider the induced filtration* $W_\bullet$ *on* W *defined by* $W_i = W \cap V_i$ *and denote by* m *the integer satisfying* $W_m \neq 0$ *and* $W_{m+1} = 0$*. Then*

(i) *there is an inclusion* $W_0/W_1 \hookrightarrow Q$ *and the connection* ∇ *induces sheaf inclusions*

$$W_i/W_{i+1} \hookrightarrow (W_{i-1}/W_i) \otimes \Omega_X^1.$$

(ii) *we have a decreasing sequence of integers*

$$\mathrm{rk}(Q) = q \geq r_0 \geq r_1 \geq \cdots \geq r_m \geq 1 \qquad \textit{and} \qquad \sum_{i=0}^{m} r_i = \mathrm{rk}(W). \tag{3.4.4}$$

with $r_i = \mathrm{rk}(W_i/W_{i+1})$ *for* $0 \leq i \leq m$*.*

Proof. Since W is ∇-invariant, the connection ∇ induces the horizontal maps of the following commutative diagram

$$\begin{array}{ccc} W_i/W_{i+1} & \longrightarrow & (W_{i-1}/W_i)\otimes\Omega^1_X \\ \downarrow & & \downarrow \\ V_i/V_{i+1} & \longrightarrow & (V_{i-1}/V_i)\otimes\Omega^1_X \end{array}$$

Since the vertical arrows are inclusions and the lower horizontal map is an isomorphism (by Definition 3.1.1), we obtain part (i). Part (ii) follows immediately from part (i). ☐

Combining the inclusions of Lemma 3.4.3 (i), we obtain for any $0 \leq i \leq m$

$$W_i/W_{i+1} \hookrightarrow Q \otimes (\Omega^1_X)^{\otimes i}.$$

Hence, if Q is semistable, we have the inequality $\mu(W_i/W_{i+1}) \leq \mu(Q) + i(2g-2)$ and we obtain

$$\deg(W) = \sum_{i=0}^{m} r_i \mu(W_i/W_{i+1}) \leq \mathrm{rk}(W)\mu(Q) + (2g-2)\left(\sum_{i=0}^{m} i r_i\right)$$

or, equivalently,

$$\mu(W) \leq \mu(Q) + \frac{2g-2}{\mathrm{rk}(W)}\left(\sum_{i=0}^{m} i r_i\right)$$

By relation (3.1.4) we have $\mu(V) = \mu(Q) + (l-1)(g-1)$, so the semistability condition $\mu(W) \leq \mu(V)$ will be implied by the inequality

$$2\left(\sum_{i=0}^{m} i r_i\right) \leq \mathrm{rk}(W)(l-1).$$

Obviously the length m of the induced filtration on W is bounded by $l-1$, hence it suffices to show that $2\left(\sum_{i=0}^{m} i r_i\right) \leq \mathrm{rk}(W)m$. But the latter inequality reduces to

$$\mathrm{rk}(W)m - 2\left(\sum_{i=0}^{m} i r_i\right) = \sum_{i=0}^{m}(m-2i)r_i = \sum_{i=0}^{[\frac{m}{2}]}(m-2i)(r_i - r_{m-i}) \geq 0,$$

which holds because of the inequalities (3.4.4).

Finally, if Q is stable, then one easily deduces from the preceding inequalities that equality $\mu(W) = \mu(V)$ holds if and only if $W = V$. Hence (V, ∇) is stable. ☐

Although we will not use the next result, we mention an interesting corollary (due to X. Sun)

Corollary 3.4.5 ([Su2] Theorem 2.2)**.** *Let E be a vector bundle over X. If E is semistable (resp. stable), then the direct image under the Frobenius morphism $F_*(E)$ is semistable (resp. stable).*

Proof. It suffices to apply Proposition 3.4.2 to the oper $(F^*(F_*(E)), \nabla^{can}, V_\bullet)$, where $V_\bullet$ is the canonical filtration, which satisfies $V_0/V_1 = E$ (see Example 3.1.6). ☐

Remark 3.4.6. *We note that Proposition 3.4.2 and Lemma 3.4.3 are essentially a reformulation of X. Sun's arguments in the set-up of opers.*

4. Quot-schemes and Frobenius-destabilized vector bundles

4.1. **Statement of the results.** Let $r \geq 2$ be an integer and put

$$C(r,g) = r(r-1)(r-2)(g-1).$$

The purpose of this section is to show the following

Theorem 4.1.1. *Let X be a smooth, projective curve of genus $g \geq 2$ over an algebraically closed field k of characteristic $p > 0$. If $p > C(r,g)$, then we have*

(1) *Every stable and Frobenius-destabilized vector bundle V of rank r and slope $\mu(V) = \mu$ over X is a subsheaf $V \hookrightarrow F_*(Q)$ for some stable vector bundle Q of rank $\mathrm{rk}(Q) < r$ and $\mu(Q) < p\mu$.*

(2) *Conversely, given a semistable vector bundle Q with $\mathrm{rk}(Q) < r$ and $\mu(Q) < p\mu$, every subsheaf $V \hookrightarrow F_*(Q)$ of rank $\mathrm{rk}(V) = r$ and slope $\mu(V) = \mu$ is semistable and destabilized by Frobenius.*

In the case when the degree of the Frobenius-destabilized bundle equals 0, we will prove the following refinement of Theorem 4.1.1.

Theorem 4.1.2. *Let X be a smooth, projective curve of genus $g \geq 2$ over an algebraically closed field k of characteristic p. If $p > C(r,g)$, then every stable and Frobenius-destabilized vector bundle V of rank r and of degree 0 over X is a subsheaf*

$$V \hookrightarrow F_*(Q)$$

for some stable vector bundle Q of rank $\mathrm{rk}(Q) < r$ and degree $\deg(Q) = -1$.

4.2. **Proof of Theorem 4.1.1.** Let V be a stable and Frobenius-destabilized vector bundle of rank r and slope $\mu(V) = \mu$. Consider the first quotient Q of the Harder-Narasimhan filtration of $F^*(V)$. If Q is not stable, we replace Q by a stable quotient. This shows the existence of a stable vector bundle Q such that

$$F^*(V) \to Q \qquad \text{and} \qquad p\mu = \mu(F^*(V)) > \mu(Q).$$

Moreover $\mathrm{rk}(Q) < \mathrm{rk}(V) = r$. By adjunction we obtain a non-zero map

$$V \to F_*(Q).$$

Thus to prove the Theorem 4.1.1(1) it will suffice to prove that this map will be an injection.

Suppose that this is not the case. Then the image of $V \to F_*(Q)$ generates a subbundle, say, $W \subset F_*(Q)$ and one has $1 \leq \mathrm{rk}(W) \leq r-1$ and by the stability of V, we have

$$\mu(V) = \mu < \mu(W).$$

Now we observe that we can bound $\mu(W)$ from below

$$\mu(W) \geq \mu + \frac{1}{r(r-1)} > \frac{\mu(Q)}{p} + \frac{1}{r(r-1)}.$$

The proof of Theorem 4.1.1(1) will now follow from Proposition 4.2.1 below applied with $\delta = \frac{1}{r(r-1)}$ and $n = r-1$.

Let us also indicate how to deduce Theorem 4.1.1(2) from Proposition 4.2.1 as well. Let $V \subset F_*(Q)$ be a subsheaf of $\mathrm{rk}(V) = r$ and $\mu(V) = \mu$ with $\mu(Q)/p \leq \mu$. If V is not semi-stable, then there exists a subsheaf $W \subset V$ with $\mu(W) > \mu(V)$ and, more precisely, $\mu(W) \geq \mu(V) + \frac{1}{r(r-1)}$. Moreover $W \subset V \subset F_*(Q)$. Replacing W by a subbundle generated by it in $F_*(Q)$, we see that

we have a subbundle $W \subset F_*(Q)$ with $\mu(W) \geq \frac{\mu(Q)}{p} + \frac{1}{r(r-1)}$. We can now apply Proposition 4.2.1 with $\delta = \frac{1}{r(r-1)}$ and $n = r-1$ to obtain a contradiction. Hence V is semistable. The fact that V is Frobenius-destabilized follows immediately by adjunction. This completes the proof of Theorem 4.1.1.

Proposition 4.2.1. *Let Q be a semistable vector bundle over the curve X. Let $\delta \in \mathbb{R}_+^*$ and let n be a positive integer. Assume that $p > \frac{(n-1)(g-1)}{\delta}$. Then any subbundle $W \subset F_*(Q)$ of rank $\mathrm{rk}(W) \leq n$ has slope*

$$\mu(W) < \frac{\mu(Q)}{p} + \delta.$$

Proof. We recall from Example 3.1.6 that the triple $(F^*(F_*(Q)), \nabla^{can}, V_\bullet)$ is a dormant oper of type $q = \mathrm{rk}(Q)$. Let $W \subset F_*(Q)$ be a subbundle. Then let $W_0 = F^*(W) \subset F^*(F_*(Q)) = V$ and consider the induced flag on W_0, i.e. $W_i = W_0 \cap V_i$. We now apply Lemma 3.4.3 (i) to the subbundle $W_0 \subset V$ and, using the notation of that lemma, we have

$$q \geq r_0 \geq r_1 \geq \cdots \geq r_m \geq 1 \qquad \text{and} \qquad \sum_{i=0}^{m} r_i = \mathrm{rk}(W) = w. \tag{4.2.2}$$

It is shown in [Su2] formula (2.12) that

$$\mu(F_*(Q)) \geq \mu(W) + \frac{2(g-1)}{pw} \sum_{i=0}^{m} \left(\frac{p-1}{2} - i \right) r_i. \tag{4.2.3}$$

We will bound the sum on the right from below to obtain a bound on $\mu(F_*(Q)) - \mu(W)$. This will be a substantial strengthening of the results of [JRXY], [LanP] and [J1].

We have the equalities

$$\begin{aligned} \sum_{i=0}^{m} \left(\frac{p-1}{2} - i \right) r_i &= \sum_{i=0}^{m} \frac{p-1}{2} r_i - \sum_{i=0}^{m} i r_i \\ &= \frac{p-1}{2} \sum_{i=0}^{m} r_i - \sum_{i=0}^{m} i r_i, \\ &= w \frac{p-1}{2} - \sum_{i=0}^{m} i r_i. \end{aligned}$$

So to bound the expression on the right from below, it will suffice to bound the sum $\sum_{i=0}^{m} i r_i$ from above subject to constraints (4.2.2). This is done in the following

Lemma 4.2.4. *For any sequence of positive integers $r_\bullet = (r_0, r_1, \ldots, r_m)$ satisfying the constraints (4.2.2) we have the inequality*

$$S(r_\bullet) = \sum_{i=0}^{m} i r_i \leq \frac{w(w-1)}{2}.$$

Moreover, we have equality if and only if $m = w-1$ and $r_0 = \cdots = r_m = 1$.

Proof. Given a sequence $r_\bullet$ we introduce the sequence of integers $s_\bullet = (s_0, \ldots, s_{m+1})$ with $s_i \geq 0$ defined by the relations

$$r_i = 1 + s_{m+1} + s_m + \ldots + s_{i+1} \qquad \text{for } 0 \leq i \leq m, \qquad \text{and} \qquad r_0 + s_0 = q.$$

We compute $S(r_\bullet) = \frac{m(m+1)}{2} + \sum_{i=1}^{m+1} \frac{i(i-1)}{2} s_i$. The problem is then equivalent to determine an upper bound for $S(r_\bullet)$, where the integers $s_\bullet$ are subject to the two constraints $\sum_{i=0}^{m+1} s_i = q-1$ and $\sum_{i=1}^{m+1} is_i = w-(m+1)$. Since $s_i \geq 0$ and $\frac{i(i-1)}{2} \leq \frac{im}{2}$ for $1 \leq i \leq m+1$, we obtain the inequality

$$\sum_{i=1}^{m+1} \frac{i(i-1)}{2} s_i \leq \sum_{i=1}^{m+1} \frac{m}{2} is_i = \frac{m}{2}(w-(m+1)),$$

hence $S(r_\bullet) \leq \frac{mw}{2}$. If we vary m in the range $0 \leq m \leq w-1$, we observe that the maximum is obtained for $m = w-1$ and that equality holds for $s_1 = \ldots = s_{m+1} = 0$, $s_0 = q-1$, i.e. $r_0 = \cdots = r_m = 1$. ☐

Combining the previous lemma with inequality (4.2.3), we obtain

$$\mu(F_*(Q)) \geq \mu(W) + \frac{2(g-1)}{pw}\left[w\frac{p-1}{2} - \frac{w(w-1)}{2}\right] = \mu(W) + (g-1)\left(1 - \frac{w}{p}\right).$$

As by Lemma 2.2.1 we have $\mu(F_*(Q)) = \frac{\mu(Q)}{p} + (g-1)(1-\frac{1}{p})$, we obtain the inequality

$$\mu(W) \leq \frac{\mu(Q)}{p} + (g-1)\left(\frac{w-1}{p}\right).$$

Hence if $\frac{(g-1)(w-1)}{p} < \delta$, or equivalently $p > \frac{(w-1)(g-1)}{\delta}$, we obtain the desired inequality. ☐

4.3. **Proof of Theorem 4.1.2.** Let V be a Frobenius-destabilized vector bundle of degree 0 and rank r. We will need a lemma.

Lemma 4.3.1. *Let E be a non-semistable vector bundle with* $\deg E = 0$*. Then there exists a semistable vector bundle Q with* $\deg(Q) = -1$ *and* $\mathrm{rk}(Q) < \mathrm{rk}(E)$ *and such that* $\mathrm{Hom}(E, Q) \neq 0$*.*

Proof. It suffices to prove the existence of a semistable subsheaf $S \subset V = E^*$ with $\deg S = 1$ and then take the dual. We will prove that by a double induction. Given two positive integers r and d we introduce the induction hypothesis $H(r,d)$: "If V contains a semistable subsheaf of rank r and degree d, then V contains a semistable subsheaf of degree 1".

We will show that $H(r,d)$ holds for any $1 \leq r < \mathrm{rk}(E)$ and any $d \geq 1$. We first observe that $H(r,1)$ holds trivially for any r, and $H(1,d)$ holds for any d, because there exist degree 1 and rank 1 subsheaves of any degree d line bundle.

We now suppose that $H(r,d)$ holds for any pair (r,d) with $1 \leq r < r_0$ and we will deduce that $H(r_0,d)$ holds for any $d \geq 1$. As observed above $H(r_0,1)$ holds. So assume that $H(r_0,d)$ holds. We will show that $H(r_0,d+1)$ also holds. So assume that V contains a semistable subsheaf W of rank r and degree $d+1$. Two cases can occur[2]: either a general negative elementary transform $W' \subset W$ of colength one is semistable or any such W' is non semistable. In the first case we can take a semistable elementary transform W', which has degree d, and apply $H(r_0,d)$, which holds by induction. In the second case we pick any elementary transform W' and consider the first piece $M \subset W'$ of its Harder-Narasimhan filtration. Then it follows from the definition of M that

$$\frac{d}{r_0} = \mu(W') < \mu(M) \qquad \text{and} \qquad \mathrm{rk}(M) < r_0.$$

In particular, $\deg(M) \geq 1$. So we can apply $H(\mathrm{rk}(M), \deg(M))$, which holds by induction, and we are done. ☐

[2]We are grateful to Peter Newstead for having pointed out this fact.

We apply the lemma to the non-semistable bundle $V = F^*(E)$ and obtain a non-zero map

$$F^*(E) \to Q \qquad \text{with} \qquad \deg(Q) = -1,\ \mathrm{rk}(Q) \leq r-1.$$

We now continue as in the proof of Theorem 4.1.1(1) to show that the map $E \to F_*(Q)$ obtained by adjunction is injective.

4.4. **Loci of Frobenius-destabilized vector bundles.** Let $\mathcal{M}(r)$ denote the coarse moduli space of S-equivalence classes of semistable bundles of rank r and degree 0 over the curve X. Let

$$\mathcal{J}(r) \subset \mathcal{M}(r)$$

be the locus of semistable bundles E which are destabilized by Frobenius pull-back, i.e. $F^*(E)$ is not semistable. Set-theoretically the locus $\mathcal{J}(r)$ is well-defined, since, given a strictly semistable bundle E with associated graded $\mathrm{gr}(E) = E_1 \oplus \cdots \oplus E_l$ with E_i stable, one observes that E is Frobenius-destabilized if and only if at least one of the stable summands E_i is Frobenius-destabilized. Moreover, $\mathcal{J}(r)$ is a closed subvariety of $\mathcal{M}(r)$. Let $\mathcal{J}^s(r) \subset \mathcal{J}(r)$ be the open subset corresponding to stable bundles.

Let $1 \leq q \leq r-1$ be an integer and let $\mathcal{M}(q,-1)$ be the moduli space of semistable bundles of rank q and degree -1 over the curve X. As $\gcd(q,-1) = 1$ we are in the coprime case and so every semistable bundle $Q \in \mathcal{M}(q,-1)$ is stable. In particular, we see that there exists a universal Poincaré bundle $\mathcal{U}$ on $\mathcal{M}(q,-1) \times X$. Let

$$\alpha : \mathcal{Q}\mathrm{uot}(q,r,0) := \mathrm{Quot}^{r,0}((F \times \mathrm{id}_{\mathcal{M}(q,-1)})_* \mathcal{U}) \longrightarrow \mathcal{M}(q,-1)$$

be the relative Quot-scheme [Gr] over $\mathcal{M}(q,-1)$. The fibre $\alpha^{-1}(Q)$ over a point $Q \in \mathcal{M}(q,-1)$ equals $\mathrm{Quot}^{r,0}(F_*(Q))$. We note that $\mathcal{Q}\mathrm{uot}(q,r,0)$ is a proper scheme [Gr].

We are now ready to restate Theorems 4.1.2 and 4.1.1(2) in a geometrical set-up.

Theorem 4.4.1. *If $p > C(r,g)$, then the image of the forgetful morphism*

$$\pi : \coprod_{q=1}^{r-1} \mathcal{Q}\mathrm{uot}(q,r,0) \longrightarrow \mathcal{M}(r), \qquad [E \subset F_*(Q)] \mapsto E$$

is contained in the locus $\mathcal{J}(r)$ and contains the closure $\overline{\mathcal{J}^s(r)}$ of the stable locus $\mathcal{J}^s(r)$.

Question 4.4.2. *For $r \geq 3$, is the locus $\mathcal{J}(r)$ equal to the closure $\overline{\mathcal{J}^s(r)}$ in $\mathcal{M}(r)$? In other words, does any irreducible component of $\mathcal{J}(r)$ contain stable bundles?*

4.5. **Maximal degree of subbundles of $F_*(Q)$.** The next proposition will not be used in this paper.

Proposition 4.5.1. *Let Q be a semistable vector bundle of rank $q = \mathrm{rk}Q$ satisfying*

$$q < r < pq \qquad \text{and} \qquad -(r-q)(g-1) \leq \deg(Q) < 0.$$

If $p > r(r-1)(g-1)$, then the maximal degree of rank-r subbundles of $F_*(Q)$ equals 0. In particular, any subsheaf of $F_*(Q)$ of degree 0 is a subbundle.

Proof. By Proposition 2.3.2 we know that $\mathrm{Quot}^{r,0}(F_*(Q)) \neq \emptyset$, hence the maximal degree of rank-r subbundles is at least 0. It is therefore enough to show that any rank-r subbundle $W \subset F_*(Q)$ satisfies $\mu(W) \leq 0$. We apply Proposition 4.2.1 with $n = r$ and $\delta = \frac{1}{r}$, which leads to $\mu(W) < \frac{1}{r}$, hence $\mu(W) \leq 0$. □

5. Harder-Narasimhan polygons of local systems

5.1. **Harder-Narasimhan filtration.** Given a vector bundle V over X we consider its Harder-Narasimhan filtration

$$V_\bullet^{HN}: \qquad 0 = V_l \subsetneq V_{l-1} \subsetneq \cdots \subsetneq V_1 \subsetneq V_0 = V$$

and we denote for $1 \leq i \leq l$ the slopes of the successive quotients $\mu_i = \mu(V_{i-1}/V_i)$, which satisfy

$$\mu_l > \mu_{l-1} > \cdots > \mu_2 > \mu_1.$$

We will also use the notation

$$\mu_{max}(V) = \mu_l \qquad \text{and} \qquad \mu_{min}(V) = \mu_1.$$

Then the numerical information $(\mathrm{rk}(V_i), \deg(V_i))$ associated to the Harder-Narasimhan flag can be conveniently organized into a convex polygon in the plane $\mathbb{R}^2$, denoted by $\mathscr{P}_V$. It is the convex polygon with vertices (or "break points") at the points $(\mathrm{rk}(V_i), \deg(V_i))$ for $0 \leq i \leq l$. The segment joining $(\mathrm{rk}(V_i), \deg(V_i))$ and $(\mathrm{rk}(V_{i-1}), \deg(V_{i-1}))$ has slope μ_i.

We will use the following result (see [LasP1] Lemma 4.2, [Su1] Theorem 3.1 or [She] Corollary 2)

Lemma 5.1.1. *If the local system (V, ∇) is semistable, then we have for $1 \leq i \leq l-1$*

$$\mu_{i+1} - \mu_i \leq 2g - 2.$$

Remark 5.1.2. *Note that [Su1] and [She] make the assumption that $\psi(V, \nabla) = 0$. In fact, this assumption is not needed in their proofs.*

5.2. **A theorem of Shatz.** The set of convex polygons in the plane $\mathbb{R}^2$ starting at $(0,0)$ and ending at $(r, 0)$ is partially ordered: if $\mathscr{P}_1, \mathscr{P}_2$ are two such polygons, we say that $\mathscr{P}_1 \succcurlyeq \mathscr{P}_2$ if $\mathscr{P}_1$ lies on or above $\mathscr{P}_2$. We recall the following well-known theorem by Shatz [Sha]

Theorem 5.2.1. (1) *Let V be a family of vector bundles on $X \times T$ parameterized by a scheme T of finite type. For $t \in T$, we denote by $V_t = V|_{X \times_T k(t)}$ the restriction of V to the fibre over t. For any convex polygon $\mathscr{P}$ the set*

$$S_{\mathscr{P}} = \{t \in T \mid \mathscr{P}_{V_t} \succcurlyeq \mathscr{P}\}$$

is closed in T and the union of the subsets $S_{\mathscr{P}}$ gives a stratification of T.

(2) *Let R be a discrete valuation ring and let V be a family of vector bundles parameterized by* $\mathrm{Spec}(R)$*, then*

$$\mathscr{P}_{V_s} \succcurlyeq \mathscr{P}_{V_\eta},$$

where s and η denote the closed and generic point of $\mathrm{Spec}(R)$ *respectively.*

5.3. **Oper-polygons are maximal.** Let $(V, \nabla, V_\bullet)$ be an oper of rank r, degree 0 and type 1. Then it follows from (3.1.3) and (3.1.4) that $l = r$ and for $0 \leq i \leq r$

$$\mathrm{rk}(V_i) = r - i, \qquad \deg(V_i) = i(r-i)(g-1).$$

We introduce the oper-polygon

$$\mathscr{P}_r^{oper}: \qquad \text{with vertices } \ (i, i(r-i)(g-1)) \text{ for } 0 \leq i \leq r.$$

In the next section we will see that the oper-polygon actually appears as a Harder-Narasimhan polygon of some semistable local system. Our next result says that oper-polygons are maximal among Harder-Narasimhan polygons of semistable local systems.

Theorem 5.3.1. *Let (V, ∇) be a semistable local system of rank r and degree 0. Then*

(1) *We have the inequality*
$$\mathscr{P}_r^{oper} \gneqq \mathscr{P}_V.$$
(2) *The equality*
$$\mathscr{P}_r^{oper} = \mathscr{P}_V$$
holds if and only if the triple $(V, \nabla, V_\bullet^{HN})$ *is an oper.*

Proof. Let (V, ∇) be a semistable local system of rank r and degree 0. Assume that V is not semistable. Let
$$0 = V_l \subsetneq V_{l-1} \subsetneq \cdots \subsetneq V_1 \subsetneq V_0 = V$$
be the Harder-Narasimhan filtration of V. We denote $n_i = \mathrm{rk}(V_{i-1}/V_i)$ for $1 \leq i \leq l$, so that
$$n_1 + n_2 + \cdots + n_l = r \qquad \text{and} \qquad \mathrm{rk}(V_i) = n_l + n_{l-1} + \cdots + n_{i+1}.$$
So in order to show the first part of the theorem, we have to prove that
$$\deg(V_i) \leq (g-1)(n_1 + \cdots + n_i)(n_{i+1} + \cdots + n_l) \tag{5.3.2}$$
holds for every $1 \leq i \leq l-1$. We also denote $\delta_i = \deg(V_{i-1}/V_i)$, so that $\mu_i = \mu(V_{i-1}/V_i) = \frac{\delta_i}{n_i}$,
$$\delta_1 + \delta_2 + \cdots + \delta_l = 0 \qquad \text{and} \qquad \deg(V_i) = \delta_l + \delta_{l-1} + \cdots + \delta_{i+1}.$$
The inequalities of Lemma 5.1.1 give for $1 \leq i \leq l-1$
$$(\mathcal{E}_i): \qquad \frac{\delta_{i+1}}{n_{i+1}} \leq \frac{\delta_i}{n_i} + 2g - 2.$$
We will prove (5.3.2) by a decreasing induction on i. The first step is to establish that
$$\delta_l = \deg(V_{l-1}) \leq (g-1)(n_1 + \cdots + n_{l-1})n_l. \tag{5.3.3}$$
We consider for $k = l-1, \ldots, 1$ the inequality $(\mathcal{E}_{l-1}) + (\mathcal{E}_{l-2}) + \cdots + (\mathcal{E}_k)$ multiplied by n_k:
$$\begin{array}{rcl} \dfrac{n_{l-1}}{n_l}\delta_l & \leq & \delta_{l-1} + n_{l-1}(2g-2), \\ \dfrac{n_{l-2}}{n_l}\delta_l & \leq & \delta_{l-2} + 2n_{l-2}(2g-2), \\ \vdots & \leq & \vdots \\ \dfrac{n_1}{n_l}\delta_l & \leq & \delta_1 + (l-1)n_1(2g-2). \end{array}$$
Now using $\sum_{i=1}^l \delta_i = 0$ and adding up all these equations and simplifying we get:
$$\begin{array}{rcl} \dfrac{n_1 + \cdots + n_{l-1}}{n_l}\delta_l & \leq & -\delta_l + 2(n_{l-1} + 2n_{l-2} + \cdots + (l-1)n_1)(g-1), \\ \delta_l & \leq & n_l(g-1)(\dfrac{2}{r}(n_{l-1} + \cdots + (l-1)n_1). \end{array}$$
Thus it remains to show that
$$2(n_{l-1} + 2n_{l-2} + \cdots + (l-1)n_1) \leq r(n_1 + n_2 + \cdots + n_{l-1}).$$
We introduce $m_i = n_i - 1 \geq 0$, and the previous inequality can be written as follows:
$$2(m_{l-1} + 2m_{l-2} + \cdots + (l-1)m_1 + (1 + 2 + \cdots + (l-1))) \leq (l + m_1 + \cdots + m_l)((l-1) + m_1 + \cdots + m_{l-1}),$$

and after some simplification using $1+2+\cdots+(l-1)=\frac{l(l-1)}{2}$ we get

$$\begin{aligned}2(m_{l-1}+2m_{l-2}+\cdots+(l-1)m_1) &\leq l(m_1+\cdots+m_{l-1})+(l-1)(m_1+\cdots+m_l)\\ &\quad +(m_1+\cdots+m_{l-1})(m_1+\cdots+m_l).\end{aligned}$$

After a rearrangement we are reduced to proving

$$2(m_{l-1}+2m_{l-2}+\cdots+(l-1)m_1)\leq(2l-1)(m_1+\cdots+m_{l-1})+\text{non-negative terms}.$$

Now clearly this holds as the inequality

$$2(m_{l-1}+2m_{l-2}+\cdots+(l-1)m_1)\leq(2l-1)(m_1+\cdots+m_{l-1})$$

visibly holds for all $l\geq 2$ as $m_i\geq 0$ and each term on the left is less than or equal to the corresponding term on the right. This establishes (5.3.3).

Now we will proceed by induction on i. So assume that if we have, for some i, the inequality

$$\deg(V_i)=\delta_l+\cdots+\delta_{i+1}\leq(g-1)(n_1+\cdots+n_i)(n_{i+1}+\cdots+n_l). \tag{5.3.4}$$

Then we claim that we also have the corresponding inequality for $i-1$. The new inequality which needs to be established is

$$\deg(V_{i-1})=\delta_l+\cdots+\delta_{i+1}+\delta_i\leq(g-1)(n_1+\cdots+n_{i-1})(n_i+\cdots+n_l). \tag{5.3.5}$$

We begin with the following inequalities:

$$\begin{aligned}\delta_i &= \delta_i\\ \frac{n_{i-1}}{n_i}\delta_i &\leq \delta_{i-1}+n_{i-1}(2g-2)\\ \frac{n_{i-2}}{n_i}\delta_i &\leq \delta_{i-2}+2n_{i-2}(2g-2)\\ \vdots &\leq \vdots\\ \frac{n_1}{n_i}\delta_i &\leq \delta_1+(i-1)n_1(2g-2).\end{aligned}$$

obtained by doing the following operations $n_k((\mathcal{E}_{i-1})+\cdots+(\mathcal{E}_k))$ for $k=i-1,\ldots,1$. Now we add all the inequalities for $k=i-1,\ldots,1$ and the equality $\delta_i=\delta_i$ and we obtain:

$$\frac{n_1+\cdots+n_i}{n_i}\delta_i\leq(\delta_1+\cdots+\delta_i)+(g-1)2(n_{i-1}+2n_{i-2}+\cdots+(i-1)n_1).$$

Multiplying this by $\frac{n_i}{n_1+\cdots+n_i}$ and using $\delta_1+\cdots+\delta_i=-(\delta_{i+1}+\cdots+\delta_l)$ we obtain

$$\delta_i+\frac{n_i}{n_1+\cdots+n_i}(\delta_{i+1}+\cdots+\delta_l)\leq(g-1)2\frac{n_i(n_{i-1}+2n_{i-2}+\cdots+(i-1)n_1)}{n_1+\cdots+n_i}.$$

Next we multiply the inequality (5.3.4) by $\frac{n_1+\cdots+n_{i-1}}{n_1+\cdots+n_i}$ and add to the previous inequality. We get

$$\begin{aligned}\delta_i+\delta_{i+1}+\cdots+\delta_l &\leq (g-1)(n_1+\cdots+n_{i-1})(n_{i+1}+\cdots+n_l)\\ &\quad +(g-1)2n_i\frac{n_{i-1}+2n_{i-2}+\cdots+(i-1)n_i}{n_1+\cdots+n_i}.\end{aligned}$$

So in order to show that the required inequality (5.3.5) holds, it is enough to establish that

$$2\left(\frac{n_{i-1}+2n_{i-2}+\cdots+(i-1)n_1}{n_1+\cdots+n_i}\right)\leq n_1+\cdots+n_{i-1}.$$

Equivalently we have to establish that

$$2(n_{i-1}+2n_{i-2}+\cdots+(i-1)n_1) \leq (n_1+\cdots+n_{i-1})(n_1+\cdots+n_i).$$

But this follows immediately by introducing $m_i = n_i - 1 \geq 0$ as was done in the first step. Since this is straightforward, we omit the details. This finishes the proof of the first part.

In order to show part two, we will directly check that a triple $(V, \nabla, V_\bullet^{HN})$ with $\mathscr{P}_V = \mathscr{P}_r^{oper}$ satisfies the two conditions of Definition 3.1.1. Given an integer i with $1 \leq i \leq l-1$ we consider the $\mathcal{O}_X$-linear map ∇_i induced by the connection ∇

$$\nabla_i : V_i \longrightarrow (V/V_i) \otimes \Omega_X^1.$$

We consider the composite $\overline{\nabla}_i$ of ∇_i with the canonical projection $(V/V_i)\otimes\Omega_X^1 \to (V/V_{i-1})\otimes\Omega_X^1$. On one hand

$$\mu_{max}((V/V_{i-1})\otimes\Omega_X^1) = \mu((V_{i-2}/V_{i-1})\otimes\Omega_X^1) = \mu(V_{i-2}/V_{i-1}) + 2g-2,$$

and on the other hand

$$\mu_{min}(V_i) = \mu(V_i/V_{i+1}) = \mu(V_{i-2}/V_{i-1}) + 4g - 4 > \mu_{max}((V/V_{i-1})\otimes\Omega_X^1).$$

The last inequality implies that $\overline{\nabla}_i = 0$, hence $\nabla(V_i) \subset V_{i-1}\otimes\Omega_X^1$ for $1 \leq i \leq l-1$.

Since the local system (V,∇) is semistable, the map ∇_i is nonzero. Moreover, since $\nabla(V_{i+1}) \subset V_i \otimes \Omega_X^1$, the map ∇_i factorizes as follows

$$\tilde{\nabla}_i : V_i/V_{i+1} \longrightarrow (V_{i-1}/V_i)\otimes\Omega_X^1.$$

Since both sides are line bundles of the same degree, we conclude that $\tilde{\nabla}_i$ is an isomorphism. $\square$

5.4. **Correspondence between dormant opers and Quot-schemes.** In this section we will determine all semistable local systems (V,∇) with $\mathscr{P}_V = \mathscr{P}_r^{oper}$ and $\psi(V,\nabla) = 0$.

Theorem 5.4.1. *Let $r \geq 2$ be an integer and assume $p > C(r,g)$. Then we have*

1. *Given a line bundle Q of degree $\deg(Q) = -(r-1)(g-1)$, the Quot-scheme $\mathrm{Quot}^{r,0}(F_*(Q))$ is non-empty and any vector bundle $W \in \mathrm{Quot}^{r,0}(F_*(Q))$ gives under pull-back by the Frobenius morphism a semistable local system*
$$(F^*(W), \nabla^{can}) \qquad \textit{with} \qquad \mathscr{P}_{F^*W} = \mathscr{P}_r^{oper},$$
i.e., the triple $(F^*(W), \nabla^{can}, (F^*(W))_\bullet^{HN})$ is a dormant oper.
2. *Conversely, any dormant oper of degree 0 is of the form $(F^*(W), \nabla^{can}, (F^*(W))_\bullet^{HN})$ with $W \in \mathrm{Quot}^{r,0}(F_*(Q))$ for some line bundle Q of degree $\deg(Q) = -(r-1)(g-1)$.*

Proof. First we note that the non-emptiness of $\mathrm{Quot}^{r,0}(F_*(Q))$ has been shown in Proposition 2.3.2. Secondly, by Theorem 4.1.1 (2) any vector bundle $W \in \mathrm{Quot}^{r,0}(F_*(Q))$ is semistable, hence (F^*W, ∇^{can}) is a semistable local system. Thus it remains to check that $\mathscr{P}_{F^*W} = \mathscr{P}_r^{oper}$. As in the proof of Proposition 4.2.1 we induce the oper flag $V_\bullet$ of $V = F^*(F_*(Q))$ on $W_0 = F^*(W)$. Using the same notation as in the proof of Proposition 4.2.1, we immediately deduce from the inequalities (4.2.2) that $m = r-1$ and $r_0 = \cdots = r_m = 1$. Moreover, since W_i/W_{i+1} is a subsheaf of V_i/V_{i+1}, we have the inequalities for $0 \leq i \leq m$

$$\deg(W_i/W_{i+1}) \leq \deg(V_i/V_{i+1}) = \deg(Q) + i(2g-2).$$

Summing over i we obtain

$$0 = \deg(W_0) = \sum_{i=0}^{m} \deg(W_i/W_{i+1}) \leq \sum_{i=0}^{m} \deg(V_i/V_{i+1}) = (m+1)\deg(Q) + \frac{m(m+1)}{2}(2g-2) = 0.$$

Hence we deduce that the previous inequalities are equalities $\deg(W_i/W_{i+1}) = \deg(Q)+i(2g-2)$, so that the induced flag $W_\bullet$ on $F^*(W)$ is in fact the oper flag. Moreover, since the quotients W_i/W_{i+1} are line bundles and the sequence of slopes $\mu(W_i/W_{i+1})$ is strictly increasing, the flag $W_\bullet$ coincides with the Harder-Narasimhan filtration of $F^*(W)$. This proves part one.

In order to show part two, we observe that any dormant oper is of the form $(F^*(W), \nabla^{can}, V_\bullet)$, where by Remark 3.1.5 the oper flag $V_\bullet$ is necessarily the Harder-Narasimhan filtration $(F^*(W))_\bullet^{HN}$. If we denote by Q the line bundle quotient V_0/V_1, then $\deg(Q) = -(r-1)(g-1)$, and by adjunction we obtain a non-zero map $W \to F_*(Q)$. We then conclude, as in the proof of Proposition 4.1.1, that $W \hookrightarrow F_*(Q)$ is an injection. $\square$

The previous result leads to a description of the set of all dormant opers of degree 0 as a relative Quot-scheme. We need to introduce some notation. We consider a universal line bundle $\mathcal{U}$ over $X \times \mathrm{Pic}^{-(r-1)(g-1)}(X)$ and denote by

$$\alpha : \mathcal{Q}\mathrm{uot}(r, 0) := \mathrm{Quot}^{r,0}((F \times \mathrm{id}_{\mathrm{Pic}})_*\mathcal{U}) \longrightarrow \mathrm{Pic}^{-(r-1)(g-1)}(X)$$

the relative Quot-scheme over the Picard variety $\mathrm{Pic}^{-(r-1)(g-1)}(X)$. The fibre $\alpha^{-1}(Q)$ over a line bundle $Q \in \mathrm{Pic}^{-(r-1)(g-1)}(X)$ equals the Quot-scheme $\mathrm{Quot}^{r,0}(F_*(Q))$.

The group $\mathrm{Pic}^0(X)$ parameterizing degree 0 line bundles over X acts on the relative Quot-scheme $\mathcal{Q}\mathrm{uot}(r, 0)$ via tensor product. Note that by the projection formula

$$\text{if } W \in \mathrm{Quot}^{r,0}(F_*(Q)), \qquad \text{then } W \otimes L \in \mathrm{Quot}^{r,0}(F_*(Q \otimes L^{\otimes p})).$$

Using this action one observes that the fibres $\alpha^{-1}(Q)$ are all isomorphic for $Q \in \mathrm{Pic}^{-(r-1)(g-1)}(X)$. On the other hand the determinant map

$$\det : \mathcal{Q}\mathrm{uot}(r, 0) \longrightarrow \mathrm{Pic}^0(X), \qquad W \mapsto \det W$$

gives another fibration of $\mathcal{Q}\mathrm{uot}(r, 0)$ over $\mathrm{Pic}^0(X)$ and, using again the action of $\mathrm{Pic}^0(X)$, one notices that the fibres of det are all isomorphic. We denote by

$$\mathcal{Q}\mathrm{uot}(r, \mathcal{O}_X) := \det^{-1}(\mathcal{O}_X)$$

the fibre over $\mathcal{O}_X$.

Then Theorem 5.4.1 implies the following

Proposition 5.4.2. *If $p > C(r, g)$, there is a one-to one correspondence between the set of dormant opers of rank r and degree 0 (resp. with fixed trivial determinant) and the relative Quot-scheme $\mathcal{Q}\mathrm{uot}(r, 0)$ (resp. $\mathcal{Q}\mathrm{uot}(r, \mathcal{O}_X)$).*

Remark 5.4.3. *By Proposition 3.4.2 any oper (of type 1) is stable, which implies that, if $p > C(r, g)$, any vector bundle $W \in \mathrm{Quot}^{r,0}(F_*(Q))$ is stable.*

6. Finiteness of the scheme of nilpotent PGL(r)-opers

The purpose of this section is to show that there are only a finite number of dormant opers with trivial determinant (Corollary 6.1.6). This finiteness result will imply that certain Quot-schemes have the expected dimension (Theorem 6.2.1).

6.1. **Nilpotent opers.**

Definition 6.1.1. *We say that an oper $(V, \nabla, V_\bullet)$ is nilpotent if its p-curvature $\psi(V, \nabla)$ is nilpotent.*

We remark that by Proposition 2.1.2 (ii) the property of being nilpotent is *not* invariant under tensor product by rank-1 local systems. For $\mathrm{PGL}(r)$-opers we therefore take the following

Definition 6.1.2. *We say that a $\mathrm{PGL}(r)$-oper $(\overline{V}, \overline{\nabla}, \overline{V}_\bullet)$ is nilpotent (resp. a dormant oper) if some lift $\sigma_\theta(\overline{V}, \overline{\nabla}, \overline{V}_\bullet) \in \mathfrak{Op}_{\mathrm{SL}(r)}(X)$ is nilpotent (resp. a dormant oper).*

By Proposition 3.3.3 we obtain that $(\overline{V}, \overline{\nabla}, \overline{V}_\bullet)$ is nilpotent if and only if $\mathrm{HM}(\overline{V}, \overline{\nabla}, \overline{V}_\bullet) = 0$. We will denote by $\mathrm{Nilp}_r(X) := \mathrm{HM}^{-1}(0) \subset \mathfrak{Op}_{\mathrm{PGL}(r)}(X)$ the fiber over 0 of the Hitchin-Mochizuki map. It parameterizes nilpotent $\mathrm{PGL}(r)$-opers and contains in particular Frobenius $\mathrm{PGL}(r)$-opers.

Theorem 6.1.3. *The scheme $\mathrm{Nilp}_r(X)$ is finite.*

Proof. By Proposition 3.2.1 the scheme $\mathrm{Nilp}_r(X)$ is a closed subvariety of an affine space, hence it is affine. It suffices therefore to show that it is proper over $\mathrm{Spec}(k)$. Using composition with the section $\sigma_\theta : \mathfrak{Op}_{\mathrm{PGL}(r)}(X) \to \mathfrak{Op}_{\mathrm{SL}(r)}(X)$ it will be enough to show that the substack $\mathcal{N}\mathrm{ilp}_r(X)$ of $\mathfrak{Op}_{\mathrm{SL}(r)}(X)$ defined as the fiber over 0 of the morphism $\mathfrak{Op}_{\mathrm{SL}(r)}(X) \to W_r$ (see section 3.3) is universally closed over $\mathrm{Spec}(k)$. This, in turn, will be a consequence of the following valuative criterion.

Proposition 6.1.4. *Let R be a discrete valuation ring and let s and η be the closed and generic point of $\mathrm{Spec}(R)$. For any nilpotent $\mathrm{SL}(r)$-oper $(V_\eta, \nabla_\eta, (\nabla_\eta)_\bullet)$ over $X \times \mathrm{Spec}(K)$ there exists a nilpotent $\mathrm{SL}(r)$-oper $(V_R, \nabla_R, (\nabla_R)_\bullet)$ over $X \times \mathrm{Spec}(R)$ extending $(V_\eta, \nabla_\eta, (\nabla_\eta)_\bullet)$.*

Proof. First of all we observe that the local system (V_η, ∇_η) is stable by Proposition 3.4.2. Hence we can apply [LasP1] Proposition 5.2 which asserts the existence of a local system (V_R, ∇_R) over $X \times \mathrm{Spec}(R)$ extending (V_η, ∇_η) and such that (V_s, ∇_s) is semistable and $\psi(V_s, \nabla_s)$ is nilpotent, where (V_s, ∇_s) denotes the restriction of (V_R, ∇_R) to the special fiber.

By Theorem 5.2.1 (2) the Harder-Narasimhan polygon raises under specialization, i.e. $\mathscr{P}_{V_s} \succcurlyeq \mathscr{P}_{V_\eta} = \mathscr{P}_r^{oper}$. On the other hand, by Theorem 5.3.1 (1) we have $\mathscr{P}_r^{oper} \succcurlyeq \mathscr{P}_{V_s}$ since the local system (V_s, ∇_s) is semistable. Hence we obtain equality $\mathscr{P}_{V_s} = \mathscr{P}_{V_\eta} = \mathscr{P}_r^{oper}$.

It remains to check that the extension $(V_R)_\bullet$ to $X \times \mathrm{Spec}(R)$ of the oper flag $(V_\eta)_\bullet$ has the property that the restriction to the special fiber $(V_s, \nabla_s, (V_s)_\bullet)$ is an oper. We note that by properness of the Quot-scheme the extension $(V_R)_\bullet$ exists and is unique. Restricting $(V_R)_\bullet$ to the special fiber gives a filtration by *subsheaves*

$$0 = (V_s)_r \subset (V_s)_{r-1} \subset \cdots \subset (V_s)_1 \subset (V_s)_0 = V_s \tag{6.1.5}$$

Since the degrees are constant under specialization $\deg(V_s)_i = \deg(V_\eta)_i$ for $0 \leq i \leq r-1$ and since $\mathscr{P}_{V_s} = \mathscr{P}_{V_\eta}$, we deduce that the subsheaves of the filtration (6.1.5) are subbundles, i.e. the quotients $(V_s)_i/(V_s)_{i+1}$ are torsion free. Hence the filtration (6.1.5) coincides with the Harder-Narasimhan filtration of V_s. Finally, Theorem 5.3.1 (2) allows to conclude that $(V_s, \nabla_s, (V_s)_\bullet)$ is an oper. □

This completes the proof of the theorem. □

Since dormant opers are nilpotent, we immediately obtain

Corollary 6.1.6. *There exists only a finite number of dormant opers with fixed trivial determinant.*

Remark 6.1.7. *The previous proof generalizes straightforwardly to arbitrary fibers (see [F] Theorem I.3 or [LasP1] Proposition 5.2), which implies that the Hitchin-Mochizuki morphism* HM *is proper.*

6.2. **Dimension of Quot-schemes.** We refer to section 5.4 for the correspondence between dormant opers and Quot-schemes.

Theorem 6.2.1. *Assume $p > C(r,g)$. For any line bundle Q with $\deg(Q) = -(r-1)(g-1)$ the Quot-scheme $\mathrm{Quot}^{r,0}(F_*(Q))$ is 0-dimensional.*

Proof. We deduce from Proposition 5.4.2 and Corollary 6.1.6 that $\dim \mathcal{Q}\mathrm{uot}(r, \mathcal{O}_X) = 0$. Hence $\dim \mathcal{Q}\mathrm{uot}(r, 0) = g$ and, since the isomorphism class of the fiber $\alpha^{-1}(Q)$ does not depend on Q, we obtain $\dim \mathrm{Quot}^{r,0}(F_*(Q)) = 0$. ☐

Remark 6.2.2. *In order to show that $\dim \mathrm{Quot}^{r,0}(F_*(Q)) = 0$ one can use the following shortcut: show that the natural morphism $\mathrm{Quot}^{r,0}(F_*(Q)) \to \mathfrak{Op}_{\mathrm{PGL}(r)}(X)$ is injective and use properness of the Quot-scheme. Note that, even if its proof is longer, Theorem 6.1.3 is more general and holds in any characteristic.*

7. Applications to loci of Frobenius-destabilized rank-2 vector bundles

7.1. **Dimension of any irreducible component.** In this section we will deal with rank-2 vector bundles and use the notation introduced in section 4.4. Note that in this case $C(2,g) = 0$ and that $\mathcal{J}^s(2) = \mathcal{J}(2)$ since there are no strictly semistable rank-2 Frobenius-destabilized vector bundles.

Remark 7.1.1. *It is shown in [Mo2] that $\dim \mathcal{J}(2) = 3g-4$ for a general curve X under the assumption $p > 2g-2$.*

As an application of our results on opers we obtain the following information on the locus of Frobenius-destabilized bundles $\mathcal{J}(2)$.

Theorem 7.1.2. *Any irreducible component of $\mathcal{J}(2)$ containing a dormant oper has dimension $3g-4$.*

Proof. We recall from Theorem 4.4.1 that there is a surjective morphism

$$\pi : \mathcal{Q}\mathrm{uot}(1,2,0) \longrightarrow \mathcal{J}(2),$$

and that $\alpha : \mathcal{Q}\mathrm{uot}(1,2,0) \to \mathrm{Pic}^{-1}(X)$ is a fibration, whose fibers $\alpha^{-1}(Q) = \mathrm{Quot}^{2,0}(F_*(Q))$ are all isomorphic.

We will need a more general version of Theorem 6.2.1

Lemma 7.1.3. *Let Q be a line bundle of degree $\deg(Q) = -(g-1)+d$ with $d \geq 0$. Then any irreducible component of $\mathrm{Quot}^{2,0}(F_*(Q))$ containing a dormant oper has dimension $2d$.*

Proof. We prove the result by induction on d. For $d = 0$, this is exactly Theorem 6.2.1. Consider a line bundle Q with $\deg(Q) = -(g-1)+(d+1)$ and let $\mathcal{C} \subset \mathrm{Quot}^{2,0}(F_*(Q))$ be an irreducible component containing a dormant oper $E \subset F_*(Q(-D))$, where D is an effective divisor of degree $d+1$. We decompose $D = x + D'$ with $x \in X$ and D' effective of degree d. Let $\mathcal{C}'$ be an irreducible component of $\mathrm{Quot}^{2,0}(F_*(Q(-x))) \cap \mathcal{C}$ containing E. By induction we have

$\dim \mathcal{C}' = 2d$. Moreover, $\mathrm{codim}_{\mathcal{C}}(\mathcal{C}') \leq 2$, since $\mathcal{C}' \neq \emptyset$ and $\mathcal{C}'$ can be defined as the zero scheme of a section of a rank-2 vector bundle over $\mathcal{C}$. Hence $\dim \mathcal{C} \leq 2d+2$. On the other hand, by the dimension estimate of the Quot-schemes given in Proposition 2.3.4 we have $\dim \mathcal{C} \geq 2d+2$. Therefore $\dim \mathcal{C} = 2d+2$ and we are done. □

Using the fibration α we deduce from Lemma 7.1.3 applied with $d = g-2$ that any irreducible component $\mathcal{I}$ of $\mathcal{Q}\mathrm{uot}(1,2,0)$ containing a dormant oper has dimension $3g-4$. Therefore it will suffice to show that the restriction of π to $\mathcal{I}$ is generically injective. This, in turn, will follow from the fact that a general vector bundle $E \in \mathcal{I}$ satisfies

$$f_E : F^*E \to Q \qquad \text{surjective},$$

where the map f_E is obtained by adjunction from the sheaf inclusion $E \hookrightarrow F_*(Q)$ with $Q \in \mathrm{Pic}^{-1}(X)$. In fact, if f_E is surjective, then $0 \subset \ker f_E \subset F^*E$ is the Harder-Narasimhan filtration of F^*E, which implies that the quotient Q is unique and that $\dim \mathrm{Hom}(E, F_*(Q)) = 1$.

Let us now show that the map f_E is surjective for a general $E \in \mathcal{C} \subset \mathrm{Quot}^{2,0}(F_*(Q))$, where $\mathcal{C}$ is an irreducible component containing a dormant oper and $\deg(Q) = -1$. Suppose on the contrary that this is not the case. Then any $E \in \mathcal{C}$ lies in $\mathrm{Quot}^{2,0}(F_*(Q(-x)))$ for some $x \in X$, i.e.

$$\mathcal{C} = \bigcup_{x \in X} \mathrm{Quot}^{2,0}(F_*(Q(-x))) \cap \mathcal{C}. \tag{7.1.4}$$

Two cases can occur:

(1) there exists a point $x \in X$ such that $\dim \mathrm{Quot}^{2,0}(F_*(Q(-x))) \cap \mathcal{C} = 2g-4$. Then $\mathcal{C} = \mathrm{Quot}^{2,0}(F_*(Q(-x))) \cap \mathcal{C}$ and contains a dormant oper. This contradicts Lemma 7.1.3 with $d = g-3$.

(2) for any point $x \in X$ we have $\dim \mathrm{Quot}^{2,0}(F_*(Q(-x))) \cap \mathcal{C} \geq 2g-5$. Because of (7.1.4) there exists a point $x \in X$ such that $\mathrm{Quot}^{2,0}(F_*(Q(-x))) \cap \mathcal{C}$ contains a dormant oper, contradicting again Lemma 7.1.3 with $d = g-3$. □

Remark 7.1.5. *Since the set of dormant opers is non-empty, there always exists at least one irreducible component of $\mathcal{J}(2)$ of dimension $3g-4$.*

Question 7.1.6. *Does any irreducible component of $\mathcal{J}(2)$ contain a dormant oper? The only case where the answer is known is $p = 2$: in that case by [JRXY] the locus $\mathcal{J}(2)$ is irreducible.*

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